SMC SIME
Summer Mathematics Competitions

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1st Annual

SIME

Summer Invitational Mathematics Examination
July 10, 2020 to July 24, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU HAVE STARTED YOUR TIMER.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. You will receive one point for each correct answer and no points for any blank or incorrect answers.
3. No aids other than writing utensils, erasers, blank scratch paper, blank graph paper, rulers, and compasses, are permitted. In particular, **calculators (online or handheld), smart devices, and computers are not permitted.**
4. Figures are not necessarily drawn to scale.
5. All students are eligible to participate in either the SMO or the SJMO, regardless of their score on the SIME. The SMO and SJMO will be given from July 27, 2020, to August 24, 2020.
6. You may record your answers on your test booklet, on a separate sheet of paper, or you may simulate contest environments by obtaining an AIME answer form from `https://www.maa.org/math-competitions/aime-archive` and record all of your answers, and certain other information, on the AIME answer form. The answer form will not be collected from you. Only your submission on the SIME Submission Form found at `https://forms.gle/TxNgprpCzkpoJMZz6` will be graded.

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1. Let $\triangle ABC$ be an equilateral triangle with side length 1 and M be the midpoint of side AC. Let N be the foot of the perpendicular from M to side AB. The angle bisector of angle $\angle BAC$ intersects MB and MN at X and Y, respectively. If the area of triangle $\triangle MXY$ is $\mathcal{A}$, and $\mathcal{A}^2$ can be expressed as a common fraction in the form $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m+n$.

2. Andrew rolls two fair six sided die each numbered from 1 to 6, and Brian rolls one fair 12 sided die numbered from 1 to 12. The probability that the sum of the numbers obtained from Andrew's two rolls is less than the number obtained from Brian's roll can be expressed as a common fraction in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

3. Real numbers $x, y > 1$ are chosen such that the three numbers

$$\log_4 x,\ 2\log_x y,\ \log_y 2$$

form a geometric progression in that order. If $x+y=90$, then find the value of xy.

4. Suppose that $(\underline{AB}, \underline{CD})$ is a pair of two digit positive integers (digits A and C must be nonzero) such that the product $\underline{AB} \cdot \underline{CD}$ divides the four digit number $\underline{ABCD}$. Find the sum of all possible values of the three digit number $\underline{ABC}$.

5. Let $ABCD$ be a rectangle with side lengths $\overline{AB} = \overline{CD} = 6$ and $\overline{BC} = \overline{AD} = 3$. A circle ω with center O and radius 1 is drawn inside rectangle $ABCD$ such that ω is tangent to $\overline{AB}$ and $\overline{AD}$. Suppose X and Y are points on ω that are not on the perimeter of $ABCD$ such that BX and DY are tangent to ω. If the value of XY^2 can be expressed as a common fraction in the form $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m+n$.

6. Let $P(x) = x^2 + ax + b$ be a quadratic polynomial. For how many pairs (a, b) of positive integers where $a, b < 1000$ do the quadratics $P(x+1)$ and $P(x)+1$ have at least one root in common?

7. Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ with centers $(1,1)$ and $(4,5)$ and radii $r_1 < r_2$, respectively, are drawn on the coordinate plane. The product of the slopes of the two common external tangents of $\mathcal{C}_1$ and $\mathcal{C}_2$ is 3. If the value of $(r_2 - r_1)^2$ can be expressed as a common fraction in the form $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m+n$.

8. Find the number of positive integers n between 1 and 1000, inclusive, satisfying

$$\lfloor \sqrt{n-1} \rfloor + 1 = \left\lfloor \sqrt{n + \sqrt{n} + 1} \right\rfloor$$

where $\lfloor n \rfloor$ denotes the greatest integer not exceeding n.

9. William writes the number 1 on a blackboard. Every turn, he erases the number N currently on the blackboard and replaces it with either $4N+1$ or $8N+1$ until it exceeds 1000, after which no more moves are made. If the minimum possible value of the final number on the blackboard is M, find the remainder when M is divided by 1000.

10. Consider all 2^{20} paths of length 20 units on the coordinate plane starting from point $(0,0)$ going only up or right, each one unit at a time. Each such path has a unique *bubble space*, which is the region of points on the coordinate plane at most one unit away from some point on the path. The average area enclosed by the bubble space of each path, over all 2^{20} paths, can be written as $\frac{m+n\pi}{p}$ where m, n, p are positive integers and $\gcd(m,n,p) = 1$. Find $m+n+p$.

11. Let $d_1, d_2, \ldots, d_k$ be the distinct positive integer divisors of 6^8. Find the number of ordered pairs (i,j) such that $d_i - d_j$ is divisible by 11.

12. Two sets S_1 and S_2, which are not necessarily distinct, are each selected randomly and independently from each other among the 512 subsets of $S = \{1, 2, \ldots, 9\}$. Let $\sigma(X)$ denote the sum of the elements of set X. Note that $\sigma(\emptyset) = 0$ where $\emptyset$ denotes the empty set. If $S_1 \cup S_2$ stands for the union of S_1 and S_2, the probability that $\sigma(S_1 \cup S_2)$ is divisible by 3 can be expressed as a common fraction of the form $\frac{m}{2^n}$ where m is odd and n is a positive integer. Find $m+n$.

13. In acute triangle $\triangle ABC$, $\overline{AB} = 20$ and $\overline{AC} = 21$. Let the feet of the perpendiculars from A to the angle bisectors of $\angle ACB$ and $\angle ABC$ be X and Y, respectively. Let M be the midpoint of $\overline{XY}$. Suppose P is the point on side BC such that MP is parallel to the angle bisector of $\angle BAC$. If given that $\overline{BP} = 11$, then the length of side BC can be expressed as a common fraction in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

14. Let $P(x) = x^3 - 3x^2 + 3$. For how many positive integers $n < 1000$ does there not exist a pair (a,b) of positive integers such that the equation

$$\underbrace{P(P(\ldots P}_{a \text{ times}}(x)\ldots)) = \underbrace{P(P(\ldots P}_{b \text{ times}}(x)\ldots))$$

has exactly n distinct real solutions?

15. Triangle $\triangle ABC$ has side lengths $\overline{AB} = 13, \overline{BC} = 14$, and $\overline{AC} = 15$. Suppose M and N are the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let P be a point on $\overline{MN}$, such that if the circumcircles of triangles $\triangle BMP$ and $\triangle CNP$ intersect at a second point Q distinct from P, then PQ is parallel to AB. The value of AP^2 can be expressed as a common fraction of the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.