2020 Summer MO (Day I)

July 27, 2020 to August 24, 2020

Note: For any geometry problem whose statement begins with an asterisk (∗), the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

SMO 1. The sequence of positive integers $a_0, a_1, a_2, \ldots$ is recursively defined such that a_0 is not a power of 2, and for all nonnegative integers n:

(i) if a_n is even, then a_{n+1} is the largest odd factor of a_n

(ii) if a_n is odd, then $a_{n+1} = a_n + p^2$ where p is the smallest prime factor of a_n

Prove that there exists some positive integer M such that $a_{m+2} = a_m$ for all $m \geq M$.

SMO 2. Adam has a single stack of $3 \cdot 2^n$ rocks, where n is a nonnegative integer. Each move, Adam can either split an existing stack into two new stacks whose sizes differ by 0 or 1, or he can combine two existing stacks into one new stack.

Adam keeps performing such moves until he eventually gets at least one stack with 2^n rocks. Find, with proof, the minimum possible number of times Adam could have combined two stacks.

SMO 3. $(*)$ Let $\triangle ABC$ be an acute scalene triangle with incenter I and incircle ω. Two points X and Y are chosen on minor arcs AB and AC, respectively, of the circumcircle of triangle $\triangle ABC$ such that XY is tangent to ω at P and $\overline{XY} \perp \overline{AI}$. Let ω be tangent to sides AC and AB at E and F, respectively. Denote the intersection of lines XF and YE as T. Prove that if the circumcircles of triangles $\triangle TEF$ and $\triangle ABC$ are tangent at some point Q, then lines PQ, XE, and YF are concurrent.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.