2020 Summer MO (Day II)

July 27, 2020 to August 24, 2020

Note: For any geometry problem whose statement begins with an asterisk $(*)$, the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

SMO 4. Let $p > 2$ be a fixed prime number. Find all functions $f : \mathbb{Z} \to \mathbb{Z}_p$, where the $\mathbb{Z}_p$ denotes the set $\{0, 1, \ldots, p-1\}$, such that p divides $f(f(n)) - f(n+1) + 1$ and $f(n+p) = f(n)$ for all positive integers n.

SMO 5. $(*)$ In triangle $\triangle ABC$, let E and F be points on sides AC and AB, respectively, such that $BFEC$ is cyclic. Let lines BE and CF intersect at point P, and M and N be the midpoints of $\overline{BF}$ and $\overline{CE}$, respectively. If U is the foot of the perpendicular from P to BC, and the circumcircles of triangles $\triangle BMU$ and $\triangle CNU$ intersect at second point V different from U, prove that A, P, and V are collinear.

SMO 6. We say that a number is *angelic* if it is greater than 10^{100} and all of its digits are elements of $\{1, 3, 5, 7, 8\}$. Suppose P is a polynomial with nonnegative integer coefficients such that over all positive integers n, if n is angelic, then the decimal representation of $P(s(n))$ contains the decimal representation of $s(P(n))$ as a contiguous substring, where $s(n)$ denotes the sum of digits of n.

Prove that P is linear and its leading coefficient is 1 or a power of 10.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.